

The Forcing Number of Graphs with Given Girth

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Abstract

In this paper, we study a dynamic coloring of the vertices of a graph G that starts with an initial subset S of colored vertices, with all remaining vertices being non-colored. At each discrete time interval, a colored vertex with exactly one non-colored neighbor forces this non-colored neighbor to be colored. The initial set S is called a forcing set of G if, by iteratively applying the forcing process, every vertex in G becomes colored. The forcing number, originally known as the *zero forcing number*, and denoted $F(G)$, of G is the cardinality of a smallest forcing set of G . We study lower bounds on the forcing number in terms of its minimum degree and girth, where the girth g of a graph is the length of a shortest cycle in the graph. Let G be a graph with minimum degree $\delta \geq 2$ and girth $g \geq 3$. Davila and Kenter [Theory and Applications of Graphs, Volume 2, Issue 2, Article 1, 2015] conjecture that $F(G) \geq \delta + (\delta - 2)(g - 3)$. This conjecture has recently been proven for $g \leq 6$. The conjecture is also proven when the girth $g \geq 7$ and the minimum degree is sufficiently large. In particular, it holds when $g = 7$ and $\delta \geq 481$, when $g = 8$ and $\delta \geq 649$, when $g = 9$ and $\delta \geq 30$, and when $g = 10$ and $\delta \geq 34$. In this paper, we prove the conjecture for $g \in \{7, 8, 9, 10\}$ and for all values of $\delta \geq 2$.

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1 Introduction

Graph *dynamic colorings* are graph colorings that may *change* with respect to discrete time intervals. One of the most prominent dynamic coloring is the result of the *forcing process* (originally called the *zero forcing process*), and its associated graph invariant, the *forcing number* (originally called the *zero forcing number*). These concepts first appeared during a workshop on linear algebra in relation to the *minimum rank problem* [1], and since then have been related to *domination* and *independence* [2], *network infection* [4], and *complexity* [18], to name a few. We highlight that computing the forcing number for a general graph is *NP*-hard [8], and as such, finding computationally efficient bounds in terms of easily computable graph properties is of particular interest; see, for example, [2, 6, 13, 14].

Throughout this paper all graphs will be consider simple, undirected, and finite. Let $G = (V, E)$ be a graph with order $n = |V(G)|$, and size $m = |E(G)|$. Let v be a vertex in G . A *neighbor* of v is a vertex adjacent to v . The *open neighborhood* of v in G , denoted $N_G(v)$, is the set of all neighbors of v in G , whereas the *closed neighborhood* of v in G is $N_G[v] = N_G(v) \cup \{v\}$. If the graph G is clear from the context, we simply write $N(v)$ and $N[v]$ rather than $N_G(v)$ and $N_G[v]$, respectively. For a set $S \subseteq V$, its *open neighborhood* is the set $N_G(S) = \bigcup_{v \in S} N(v)$, and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$.

We denote the *degree* of a vertex v in G by $d_G(v) = |N_G(v)|$. The minimum and maximum degrees among all vertices of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. For a subset S of vertices in G , the subgraph induced by S is denoted by $G[S]$. The degree of v in S , denoted by $d_S(v)$, is the number of neighbors of v in G that belong to S . In particular, if $S = V(G)$, then $d_S(v) = d_G(v)$. The *distance* between two vertices u and v in G , denoted $d_G(u, v)$ or simply $d(u, v)$ if the graph G is clear from context, is the minimum length of a (u, v) -path in G . The length of a shortest cycle in G is the *girth* of G , denoted by $g = g(G)$. We will use the notation P_n , C_n , K_n , and $K_{n,m}$, to denote the *path* on n vertices, the *cycle* on n vertices, the *complete graph* on n vertices, and the complete bipartite graph with partite sets of sizes n and m , respectively. If G does not contain a graph F as an induced subgraph, we say that G is *F-free*. A graph is *triangle-free* if it is K_3 -free.

Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. The *out-degree* of a vertex v in D , denoted by $\text{od}_D(v)$, is the number of vertices w such that $(v, w) \in A(D)$, where (v, w) denotes an arc from v to w ; that is, $\text{od}_D(v) = |\{w \in V(D) : (v, w) \in A(D)\}|$. We denote the number of arc in a digraph D by $m(D)$, and note that $m(D) = \sum_{v \in V(D)} \text{od}_D(v)$.

Adopting the notation of [10], given a graph G , the *forcing process* is defined as follows: Let $S \subseteq V(G)$ be an initial set of “colored” vertices; all remaining vertices being “non-colored”. A vertex in a set S , we call *S-colored*, while a vertex not in S we call *S-uncolored*. At each time step, a colored vertex v with exactly one non-colored

neighbor will change, or *force*, the non-colored neighbor to be colored. We call such a vertex v a *forcing colored vertex*, or simply a *forcing vertex*. Further, at the time when the vertex v forces its non-colored neighbor to be colored, we say that the vertex v is *played*. A set $S \subseteq V(G)$ of initially colored vertices is called a *forcing set* if, by iteratively applying the forcing process, all of $V(G)$ becomes colored. We call such a set S an *S -forcing set*. The *forcing number* of a graph G , denoted by $F(G)$, is the cardinality of a smallest forcing set. If S is a forcing set in G and v is an S -colored vertex that forces a new vertex to be colored, then we call v an *S -forcing vertex*.

2 Main Result

In this paper, we study the following intriguing conjecture posed by Davila and Kenter [11].

Conjecture 1 ([11]) *If G is a graph with girth $g \geq 3$ and minimum degree $\delta \geq 2$, then*

$$F(G) \geq \delta + (\delta - 2)(g - 3).$$

Gentner, Penso, Rautenbach, and Souzab [13] and Gentner and Rautenbach [14] have shown that Conjecture 1 is true for small girth $g \leq 6$, while Davila and Kenter [11] have proven that Conjecture 1 is true for girth $g \geq 7$ and sufficiently large minimum degree. We state these results formally as follows.

Theorem 1 *If G is a graph with girth $g \geq 3$ and minimum degree $\delta \geq 2$, then the following holds.*

- (a) ([13, 14]) *If $g \leq 6$, then Conjecture 1 is true.*
- (b) ([11]) *If $g = 7$ and $\delta \geq 481$, then Conjecture 1 is true.*
- (c) ([11]) *If $g = 8$ and $\delta \geq 649$, then Conjecture 1 is true.*
- (d) ([11]) *If $g = 9$ and $\delta \geq 30$, then Conjecture 1 is true.*
- (e) ([11]) *If $g = 10$ and $\delta \geq 34$, then Conjecture 1 is true.*

Our aim in this paper is to prove that Conjecture 1 is true when the girth $g \in \{7, 8, 9, 10\}$, for all minimum degree $\delta \geq 2$. This improves the result of Theorem 1(b), 1(c) and 1(d) which imposes a restriction on the minimum degree δ . We state our result formally as follows.

Theorem 2 *If G is a graph with girth $g \in \{7, 8, 9, 10\}$ and minimum degree $\delta \geq 2$, then Conjecture 1 is true.*

3 Known Results and Motivation

As remarked earlier, finding bounds on $F(G)$ in terms of easily computable graph properties is of interest. The earliest such bound is given in the original paper [1] which showed that $F(G)$ is at least as small as the minimum degree. In particular, we note that any initially forcing vertex must be colored along with all but one of its neighbors. We state this result formally with the following proposition.

Proposition 3 ([1]) *If G is a graph with minimum degree δ , then $F(G) \geq \delta(G)$, and this bound is sharp.*

Proposition 3 is sharp, as can be seen by considering the path P_n , the cycle C_n , and the complete graph K_n . Since the leaf of every non-trivial path is a forcing set, we note that $F(P_n) = 1$. As observed in [10], paths are the only graphs G satisfying $F(G) = 1$, and are therefore a special class of graphs when considering $F(G)$. Complete graphs have the largest possible forcing number. As observed in [10], if G is a connected graph of order $n \geq 2$, then $F(G) \leq n - 1$, with equality if and only if $G = K_n$. We remark that this upper bound of $n - 1$ implies that there exists at least one played vertex in any minimum forcing set for non-empty graphs.

Barioli et al. [3] prove that the forcing number of a graph is at least its tree-width. Using this result, and a result of Chandrana and Subramanian [7] that establishes a lower bound on the tree-width of a graph in terms of its average degree and girth, Davila and Kenter [11] prove that Conjecture 1 is true for graphs with girth at least 7 and sufficiently large minimum degree. More precisely, they prove the following result.

Theorem 4 ([11]) *Conjecture 1 is true for all graphs with minimum degree δ and given girth g , where $g \geq 7$, that satisfy*

$$\frac{(\delta - 1)^{\lfloor (g-1)/2 \rfloor - 1}}{12(g + 1)} \geq \delta + (\delta - 2)(g - 3).$$

We remark that Theorem 1(b), 1(c), 1(d) and 1(e) follows from Theorem 4 in the special case when $g \in \{7, 8, 9, 10\}$.

4 Proof of Main Result

Let G be a graph with minimum degree $\delta \geq 2$ and girth g , where $g \geq 5$, and consider a shortest cycle C in G of length g . The girth at least 5 constraint implies that no two vertices on C have a common neighbor outside C , implying that G has at least $g(\delta - 1)$ vertices. We state this observation formally as follows.

Observation 5 *If G is a graph of order n with minimum degree $\delta \geq 2$ and girth g , where $g \geq 5$, then $n \geq g(\delta - 1)$.*

In order to proceed with the proof of Theorem 2, we recall a classical result known as *Mantel's Theorem*.

Theorem 6 (Mantel's Theorem) *If a graph G on n vertices is triangle-free, then it contains at most $\frac{n^2}{4}$ edges.*

We are now ready to give the proof of Theorem 2.

Theorem 2 *If G is a graph with girth $g \in \{7, 8, 9, 10\}$ and minimum degree $\delta \geq 2$, then Conjecture 1 is true.*

Proof of Theorem 2. Let G be a graph of order n with girth g , where $g \in \{7, 8, 9, 10\}$, and minimum degree $\delta \geq 2$. We wish to show that $F(G) \geq \delta + (\delta - 2)(g - 3)$. Suppose, to the contrary, that $F(G) \leq \delta + (\delta - 2)(g - 3) - 1$. If $\delta = 2$, then $\delta + (\delta - 2)(g - 3) = \delta$, and so, by Proposition 3, $F(G) \geq \delta + (\delta - 2)(g - 3)$, a contradiction. Hence, $\delta \geq 3$. By Observation 5, $n \geq g(\delta - 1)$. Let $S \subseteq V(G)$ be a minimum forcing set of G , and so $|S| = F(G) \leq \delta + (\delta - 2)(g - 3) - 1$. We state our supposition formally as follows.

Claim 1 *The following holds.*

- (a) *If $g = 7$, then $|S| \leq 5\delta - 9$.*
- (b) *If $g = 8$, then $|S| \leq 6\delta - 11$.*
- (c) *If $g = 9$, then $|S| \leq 7\delta - 13$.*
- (d) *If $g = 10$, then $|S| \leq 8\delta - 15$.*

Let $\overline{S} = V(G) \setminus S$, and so \overline{S} is the set of all S -uncolored vertices. Thus, $|\overline{S}| = n - |S| \geq g(\delta - 1) - \delta - (\delta - 2)(g - 3) + 1 = g + (2\delta - 5) \geq g + 1$. Since S is a forcing set of G , there is a sequence x_1, x_2, \dots, x_t of played vertices in the forcing process that results in all $V(G)$ colored, where x_i denotes the forcing colored vertex played in the i th step of the process. We note that $t = |\overline{S}| > g - 2$. Let $X = \{x_1, x_2, \dots, x_{g-2}\}$. For $i \in [g - 2]$, let $X_i = N[x_i]$ and let

$$X_{\leq i} = \bigcup_{j=1}^i X_j.$$

Let S_1 be the neighbors of x_1 in S , let S_2 be the neighbors of x_2 in S that do not belong to X_1 , let S_3 be the neighbors of x_3 in S that do not belong to $X_1 \cup X_2$, and so on. Thus, $S_1 = S \cap N(x_1)$, and for $i \in [g - 2] \setminus \{1\}$, the set S_i is the set of neighbors of x_i in S that do not belong to $X_{\leq i-1}$; that is,

$$S_i = S \cap (N(x_i) \setminus X_{\leq i-1}).$$

We note that $S_i \subset X_i \setminus \{x_i\}$ for each $i \in [g-2]$. Let

$$S_X^* = \bigcup_{i=1}^{g-2} S_i \quad \text{and} \quad S_X = X \cap (S \setminus S_X^*).$$

By definition, $S_i \cap S_j = \emptyset$ for $1 \leq i, j \leq [g-2]$ and $i \neq j$, and so

$$|S_X^*| = \sum_{i=1}^{g-2} |S_i|.$$

Since the girth of G is greater than 4, the vertex x_i has at most one neighbor in X_j for each $j < i$ and $i \in [g-2]$. Let D be the digraph with vertex set $V(D) = X$ and with arc set $A(D)$ defined as follows. For each $i \in [g-2]$, we add an arc from x_i to x_j if $i > j$ and x_i has a neighbor in X_j in the graph G . We will now show a number of claims which culminate with a contradiction to G being a counterexample.

Claim 2 *The following holds.*

- (a) $|S| \geq |S_X| + |S_X^*|$.
- (b) $|S_X| \geq 1$ and $x_1 \in S_X$.
- (c) $|S_X^*| \geq (g-2)(\delta-1) - m(D)$.

Proof. Part (a) follows immediately from the fact that the sets S_X^* and S_X are vertex disjoint and $S \cap N_G[X] = S_X \cup S_X^*$. By definition of the sets S_i , $i \in [g-2]$, the first vertex played, namely x_1 , does not belong to S_X^* . However, the vertex x_1 is an S -forcing vertex and therefore belongs to S , and so $x_1 \in S_X$. This establishes Part (b). To prove part (c), we show that $|S_i| = d_G(x_i) - \text{od}_D(x_i) - 1$ for all $i \in [g-2]$. Since the vertex x_1 has exactly one S -uncolored neighbor, this implies that $|S_1| = d_G(x_1) - \text{od}_D(x_1) - 1$ noting that $\text{od}_D(x_1) = 0$. Hence we may assume that $i \geq 2$. The vertex x_i has exactly $\text{od}_D(x_i)$ neighbors in $X_{\leq i-1}$ (with at most one neighbor in X_j for each $j \in [i-1]$). Further, since x_i is a forcing vertex in the i th step of the forcing process, it has exactly one neighbor in $\bar{S} \setminus X_{\leq i-1}$. Thus, $|S_i| = d_G(x_i) - \text{od}_D(x_i) - 1$. Therefore,

$$|S_X^*| = \sum_{i=1}^{g-2} |S_i| = \sum_{i=1}^{g-2} (d_G(x_i) - \text{od}_D(x_i) - 1) \geq (g-2)(\delta-1) - m(D),$$

noting that $\sum_{i=1}^{g-2} \text{od}_D(x_i) = m(D)$. This proves Part (c). \square

Let G_D denote the underlying graph of the digraph D obtained from D by removing the directions of the arcs in $A(D)$. We note that $V(G_D) = X$ and $m(G_D) = m(D)$. Further, we note that if $x_{i_1}x_{i_2}$ is an edge in G_D , then either x_{i_1} and x_{i_2} are adjacent in G or x_{i_1} and x_{i_2} have a common neighbor in G . Let G_7 and G_8 be the graphs illustrated in Figure 1(a) and 1(b), respectively.



Figure 1: The graphs G_7 and G_8 .

Claim 3 *The following holds.*

- (a) *If $g = 7$, then $|S_X^*| \geq 5\delta - 11$. Further, if $|S_X^*| = 5\delta - 11$, then $G_D \cong K_{2,3}$.*
- (b) *If $g = 8$, then $|S_X^*| \geq 6\delta - 15$. Further, if $|S_X^*| = 6\delta - 15$, then $G_D \cong K_{3,3}$.*
- (c) *If $g = 9$, then $|S_X^*| \geq 7\delta - 15$. Further, if $|S_X^*| = 7\delta - 15$, then $G_D \cong G_7$.*
- (d) *If $g = 10$, then $|S_X^*| \geq 8\delta - 18$. Further, if $|S_X^*| = 8\delta - 18$, then $G_D \cong G_8$.*

Proof. Suppose first that $g \in \{7, 8\}$. In this case, we note that the graph G_D is triangle-free. Thus, by Mantel's Theorem, the maximum number of edges in the graph G_D , which has order $g - 2$, is $\lfloor (g - 2)^2 / 4 \rfloor$. Further, the graph $K_{\lfloor (g-2)/2 \rfloor, \lceil (g-2)/2 \rceil}$ is the unique extremal graph. This observation, together with Claim 2(c) and the fact that $m(D) = m(G_D) \leq \lfloor (g - 2)^2 / 4 \rfloor$, yields Parts (a) and (b). In particular, we note that if $g = 7$, then $m(D) \leq 6$, while if $g = 8$, then $m(D) \leq 9$.

Suppose next that $g \in \{9, 10\}$. In this case, we note that the graph G_D , which has order $g - 2$, contains neither three-cycles nor four-cycles. It is well-known (see, for example, [12], or simply use a computer) that if $g = 9$, then such a graph G_D has at most eight edges and there is a unique extremal graph of size 8, namely G_7 . Further, if $g = 10$, then such a graph G_D has at most ten edges and there is a unique extremal graph of size 10, namely G_8 . \square

Claim 4 *For $i \in [g - 2]$, the following holds.*

- (a) *If $x_i \in \overline{S}$, then x_i is adjacent in G to x_j for some $j \in [i - 1]$.*
- (b) *If x_i is adjacent in G to no vertex x_j where $j \in [i - 1]$, then $x_i \in S_X$.*

Proof. By definition of the forcing process, if $x_i \in \overline{S}$ for some $i \in [g - 2]$, then $i \geq 2$ and there exists some $j \in [i - 1]$ such that the vertex x_i is the vertex in \overline{S} that becomes colored when the vertex x_j is played, implying that x_i is adjacent to x_j in G . This proves Part (a). To prove Part (b), suppose that x_i is adjacent to no vertex x_j where $j \in [i - 1]$. By Part (a), $x_i \in S$. Since x_i is not adjacent to x_j in G for any $j \in [i - 1]$, we note that $x_i \notin S_j$. This implies, by definition of the set S_X^* , that the vertex x_i does not belong to S_X^* , implying that $x_i \in S_X$. \square

Claim 5 $g \neq 9$.

Proof. Suppose, to the contrary, that $g = 9$. As shown in the proof of Claim 3(c), the graph G_D , of order 7, contains neither three-cycles nor four-cycles and satisfies $m(G_D) \leq 8$.

Suppose that $m(G_D) = 8$, implying that $G_D \cong G_7$. By Claim 3(c), $|S_X^*| \geq 7\delta - 15$. Thus, by Claim 1 and Claim 2(a), $7\delta - 13 \geq |S| \geq |S_X| + |S_X^*| \geq |S_X| + 7\delta - 15$, implying that $|S_X| \leq 2$. By Claim 2(b), $x_1 \in S_X$. Thus for at least five values of $i \in [7] \setminus \{1\}$, the vertex x_i does not belong to S_X , and is therefore by Claim 4 adjacent to some vertex x_j where $j \in [i - 1]$. This implies that at least five edges in G_D correspond to edges in the graph G . Recall that every edge $x_{i_1}x_{i_2}$ in G_D is an edge in G or corresponds to a path $x_{i_1}vx_{i_2}$ where v is a common neighbor of x_{i_1} and x_{i_2} in G . Therefore, at least one of the two 5-cycles in G_D corresponds to a cycle in G of length less than 9, a contradiction to our supposition that $g = 9$. Hence, $m(G_D) \leq 7$.

Since $m(D) = m(G_D) \leq 7$, Claim 2(c) implies that $|S_X^*| \geq 7\delta - 14$. Thus, by Claim 1 and Claim 2(c), $7\delta - 13 \geq |S| \geq |S_X| + |S_X^*| \geq 7\delta - 13$. Hence, we must have equality throughout this inequality chain, implying that $|S| = 7\delta - 13$, and so $|S_X| = 1$ and $|S_X^*| = 7\delta - 14$. Further, $m(D) = m(G_D) = 7$ and $S_X = \{x_1\}$. Thus for all $i \in [7] \setminus \{1\}$, the vertex $x_i \notin S_X$, implying that x_i is adjacent to some vertex x_j where $j \in [i - 1]$. By the girth $g = 9$ supposition, this implies that each such vertex x_i is adjacent to exactly one vertex x_j where $j \in [i - 1]$. Therefore, G_D is a tree, and so $m(G_D) = 6$, a contradiction. \square

Claim 6 $g \neq 10$.

Proof. Suppose, to the contrary, that $g = 10$. As shown in the proof of Claim 3(d), the graph G_D , of order 8, contains neither three-cycles nor four-cycles and satisfies $m(G_D) \leq 10$.

Suppose that $m(G_D) = 10$, implying that $G_D \cong G_8$. By Claim 3(d), $|S_X^*| \geq 8\delta - 18$. Thus, by Claim 1 and Claim 2(a), $8\delta - 15 \geq |S| \geq |S_X| + |S_X^*| \geq |S_X| + 8\delta - 18$, implying that $|S_X| \leq 3$. By Claim 2(b), $x_1 \in S_X$. Thus for at least five values of $i \in [8] \setminus \{1\}$, the vertex x_i does not belong to S_X , and is therefore by Claim 4 adjacent to some vertex x_j where $j \in [i - 1]$. This implies that at least five edges in G_D correspond to edges in the graph G . Every 5-cycle in G_D corresponds to a cycle in G of length at most 10. Thus, since the girth of G is 10, every edge in a 5-cycle in G_D corresponds to a path of length 2 in G . However, every edge of G_D belongs to some 5-cycle, implying that no edge of G_D corresponds to an edge of G and therefore that $S_X = X$, and so $|S_X| = 8$, a contradiction. Hence, $m(G_D) \leq 9$.

If $m(G_D) \leq 7$, then Claim 2(c) implies that $|S_X^*| \geq 8\delta - 15$. Thus, by Claim 2, $|S| \geq |S_X| + |S_X^*| \geq 8\delta - 14$, contradicting Claim 1(d). Hence, $m(G_D) \in \{8, 9\}$, implying that G_D contains a cycle.

Since $m(D) = m(G_D) \leq 9$, Claim 2(c) implies that $|S_X^*| \geq 8\delta - 17$. Thus, by Claim 1 and Claim 2(a), $8\delta - 15 \geq |S| \geq |S_X| + |S_X^*| \geq |S_X| + 8\delta - 17$, implying that $|S_X| \leq 2$. Thus for at least six values of $i \in [8] \setminus \{1\}$, the vertex x_i does not belong to S_X , and is therefore by Claim 4 adjacent to some vertex x_j where $j \in [i - 1]$. This implies that at least six edges in G_D correspond to edges in the graph G . As observed earlier, G_D contains a cycle, but contains neither three-cycles nor four-cycles. Further, $m(G_D) \in \{8, 9\}$.

Suppose that G_D contains a 5-cycle. Since the girth of G is 10, this implies that no edge of the 5-cycle corresponds to an edge of G . Thus, there are at most $m(D) - 5 \leq 9 - 5 = 4$ edges of G_D that correspond to edges in the graph G , a contradiction. Hence, G_D has girth at least 6. Suppose that G_D contains a 6-cycle. In this case, since the girth of G is 10 at least four edges of the 6-cycle do not correspond to an edge of G . Thus, there are at most $m(D) - 4 \leq 9 - 4 = 5$ edges of G_D that correspond to edges in the graph G , a contradiction. Hence, G_D has girth at least 7, implying that $m(G_D) = 8$.

Claim 2(c) implies that $|S_X^*| \geq 8\delta - 16$. Thus, by Claim 1 and Claim 2(d), $8\delta - 15 \geq |S| \geq |S_X| + |S_X^*| \geq 8\delta - 15$. Hence, we must have equality throughout this inequality chain, implying that $|S| = 8\delta - 15$, and so $|S_X| = 1$ and $S_X = \{x_1\}$. Thus for all $i \in [8] \setminus \{1\}$, the vertex $x_i \notin S_X$, implying that x_i is adjacent to some vertex x_j where $j \in [i - 1]$. By the girth $g = 10$ supposition, this implies that each such vertex x_i is adjacent to exactly one vertex x_j where $j \in [i - 1]$. Therefore, G_D is a tree, and so $m(G_D) = 7$, a contradiction. \square

By Claims 5 and 6, $g \in \{7, 8\}$.

Claim 7 *The vertices x_1 and x_2 are not adjacent.*

Proof. Suppose, to the contrary, that the vertices x_1 and x_2 are adjacent.

Claim 7.1 *The vertex x_3 is adjacent to neither x_1 nor x_2 .*

Proof. Suppose, to the contrary, that x_3 is adjacent to x_1 or x_2 . The girth at least seven requirement implies that the vertex x_i has at most one neighbor in $X_{\leq 3}$ for every $i \in \{4, \dots, g - 2\}$. Thus, if $g = 7$, then $m(D) \leq 5$, while if $g = 8$, then, recalling that G_D is triangle-free and therefore there are at most two edges in $G_D[\{x_4, x_5, x_6\}]$, this implies that $m(D) \leq 7$. Hence, by Claim 2(c), if $g = 7$, then $|S_X^*| \geq 5\delta - 10$, while if $g = 8$, then $|S_X^*| \geq 6\delta - 13$.

Claim 7.1.1 $g = 8$.

Proof. Suppose that $g = 7$. In this case, $|S_X^*| \geq 5\delta - 10$. By Claim 2(b), $|S_X| \geq 1$. Thus, by Claim 1 and Claim 2(a), $5\delta - 9 \geq |S| \geq |S_X| + |S_X^*| \geq 5\delta - 9$. Hence, we

must have equality throughout this inequality chain, implying that $|S| = 5\delta - 9$, and so $|S_X| = 1$ and $|S_X^*| = 5\delta - 10$. Further, $S_X = \{x_1\}$. Since $x_4 \notin S_X$, the vertex x_4 is adjacent to some vertex x_j where $j \in [3]$. By the girth condition, such a vertex x_j is the only neighbor of x_4 in $X_{\leq 3}$. Since $x_5 \notin S_X$, the vertex x_5 is adjacent to some vertex x_r where $r \in [4]$. This implies by the girth seven requirement that x_5 has exactly one neighbor in $X_{\leq 4}$, and therefore that $m(D) \leq 4$. Hence, by Claim 2(c), $|S_X^*| \geq 5\delta - 9$, a contradiction. \square

By Claim 7.1.1, the girth $g = 8$. Thus, $|S_X^*| \geq 6\delta - 13$.

Claim 7.1.2 *The vertex x_4 is adjacent to no vertex x_j where $j \in [3]$.*

Proof. Suppose, to the contrary, that x_4 is adjacent to some vertex x_j , where $j \in [3]$. Such a vertex x_j is the only neighbor of x_4 in $X_{\leq 3}$. The girth eight requirement implies that each of x_5 and x_6 has at most one neighbor in $X_{\leq 4}$, and therefore that $m(D) \leq 6$. Hence, by Claim 2(c), $|S_X^*| \geq 6\delta - 12$. By Claim 2(b), $|S_X| \geq 1$. Thus, by Claim 1 and Claim 2(a), $5\delta - 11 \geq |S| \geq |S_X| + |S_X^*| \geq 6\delta - 11$. Hence, we must have equality throughout this inequality chain, implying that $|S| = 6\delta - 11$, and so $|S_X| = 1$ and $|S_X^*| = 6\delta - 12$. Further, $S_X = \{x_1\}$. Since neither x_5 nor x_6 belongs to S_X , Claim 4(b) implies that x_i is adjacent to some vertex x_j where $j \in [i-1]$ for $i \in \{5, 6\}$. This, together with the girth eight requirement, implies that the vertex x_6 has exactly one neighbor in $X_{\leq 5}$. This in turn implies that $m(D) \leq 5$. Hence, by Claim 2(c), $|S_X^*| \geq 6\delta - 11$, a contradiction. \square

By Claim 7.1.2, the vertex x_4 is adjacent to no vertex x_j where $j \in [3]$. By Claim 4(b), $x_4 \in S_X$, and so $|S_X| \geq 2$. As observed earlier, $|S_X^*| \geq 6\delta - 13$. Thus, by Claim 1 and Claim 2(a), $5\delta - 11 \geq |S| \geq |S_X| + |S_X^*| \geq 6\delta - 11$. Hence, we must have equality throughout this inequality chain, implying that $|S| = 6\delta - 11$, and so $|S_X| = 2$ and $|S_X^*| = 6\delta - 13$. Further, $S_X = \{x_1, x_4\}$.

Claim 7.1.3 *The vertex x_4 has no neighbor in $X_{\leq 3}$.*

Proof. Suppose, to the contrary, that x_4 has a neighbor in $X_{\leq 3}$. By the girth condition, this is the only neighbor of x_4 in $X_{\leq 3}$. Since neither x_5 nor x_6 belongs to S_X , Claim 4(b) implies that x_i is adjacent to some vertex x_j where $j \in [i-1]$ for $i \in \{5, 6\}$. This, together with the girth eight requirement, implies that the vertex x_5 has exactly one neighbor in $X_{\leq 4}$, and the vertex x_6 has exactly one neighbor in $X_{\leq 5}$. Therefore, $m(D) \leq 6$. Hence, by Claim 2(c), $|S_X^*| \geq 6\delta - 12$, a contradiction. \square

By Claim 7.1.3, the vertex x_4 has no neighbor in $X_{\leq 3}$, implying by our earlier observations that $m(D) \leq 6$ and therefore that $|S_X^*| \geq 6\delta - 12$, a contradiction. This completes the proof of Claim 7.1. \square

By Claim 7.1, the vertex x_3 is adjacent to neither x_1 nor x_2 . Hence by Claim 4(b), $x_3 \in S_X$, and so $|S_X| \geq 2$.

Claim 7.2 *The vertex x_3 has no neighbor in $X_{\leq 2}$.*

Proof. Suppose, to the contrary, that x_3 has a neighbor in $X_{\leq 2}$. Suppose that $g = 7$. By Claim 3, $|S_X^*| \geq 5\delta - 11$. Thus, by Claim 1 and Claim 2(a), $5\delta - 9 \geq |S| \geq |S_X| + |S_X^*| \geq 5\delta - 9$. Hence, we must have equality throughout this inequality chain, implying that $|S| = 5\delta - 9$, and so $|S_X| = 2$ and $|S_X^*| = 5\delta - 11$. Further, $S_X = \{x_1, x_3\}$. Since $x_4 \notin S_X$, the vertex x_4 is by Claim 4(b) adjacent to x_1 , x_2 or x_3 . The girth condition implies in this case that x_4 has at most one neighbor in $X_{\leq 3}$. This in turn implies that $m(D) \leq 5$. Hence, by Claim 2(c), $|S_X^*| \geq 6\delta - 10$, a contradiction. Hence, the girth $g = 8$.

The girth eight requirement implies that the vertex x_i has at most one neighbor in $X_{\leq 3}$ for every $i \in \{4, 5, 6\}$. This in turn implies that $m(D) \leq 7$. Hence, by Claim 2(c), $|S_X^*| \geq 6\delta - 13$. Thus, by Claim 1 and Claim 2(a), $5\delta - 11 \geq |S| \geq |S_X| + |S_X^*| \geq 6\delta - 11$. Hence, we must have equality throughout this inequality chain, implying that $|S| = 6\delta - 11$, and so $|S_X| = 2$ and $|S_X^*| = 6\delta - 13$. Further, $S_X = \{x_1, x_3\}$. Since $x_4 \notin S_X$, the vertex x_4 is adjacent to x_1 , x_2 or x_3 . The girth condition implies that x_4 is adjacent to exactly one vertex in $X_{\leq 3}$. Since neither x_5 nor x_6 belongs to S_X , Claim 4(b) implies that x_i is adjacent to some vertex x_j where $j \in [i - 1]$ for $i \in \{5, 6\}$. This, together with the girth eight requirement, implies that the vertex x_5 has exactly one neighbor in $X_{\leq 4}$, and the vertex x_6 has at most two neighbors in $X_{\leq 5}$. Therefore, $m(D) \leq 6$. Hence, by Claim 2(c), $|S_X^*| \geq 6\delta - 12$, a contradiction. \square

By Claim 7.2, the vertex x_3 has no neighbor in $X_{\leq 2}$. If x_4 has at most one neighbor in $X_{\leq 3}$, then $m(D) \leq 5$. If x_4 has two neighbors in $X_{\leq 3}$, then x_4 has one neighbor in $X_1 \cup X_2$ and a different neighbor in X_3 . In this case, x_5 has at most one neighbor in each of $X_1 \cup X_2$ and $X_3 \cup X_4$, implying once again that $m(D) \leq 5$. Thus if $g = 7$, then by Claim 2(c), $|S_X^*| \geq 5\delta - 10$, and so $|S| \geq |S_X| + |S_X^*| \geq 5\delta - 8$, contradicting Claim 1(a). Therefore, $g = 8$.

Claim 7.3 *The vertex x_4 is adjacent to no vertex x_j where $j \in [3]$.*

Proof. Suppose, to the contrary, that x_4 is adjacent to some vertex x_j , where $j \in [3]$. We show first that x_4 is adjacent to x_3 .

Claim 7.3.1 *x_4 is adjacent to neither x_1 nor x_2 .*

Proof. Suppose, to the contrary, that x_4 is adjacent to x_1 or x_2 . In this case, x_i has at most one neighbor in $X_1 \cup X_2 \cup X_4$ for $i \in \{5, 6\}$. Recall that G_D is triangle-free. If x_5 has

a neighbor in X_3 , then x_6 has at most one neighbor in $X_3 \cup X_5$, by the girth condition, implying that $m(D) \leq 7$. If x_5 has no neighbor in X_3 , then x_6 has at most two neighbors in $X_3 \cup X_5$, implying once again that $m(D) \leq 7$. Hence, by Claim 2(c), $|S_X^*| \geq 6\delta - 13$. Thus, by Claim 1 and Claim 2(a), $5\delta - 11 \geq |S| \geq |S_X| + |S_X^*| \geq 6\delta - 11$. Hence, we must have equality throughout this inequality chain, implying that $|S| = 6\delta - 11$, and so $|S_X| = 2$, $S_X = \{x_1, x_3\}$, and $|S_X^*| = 6\delta - 13$. Further, $m(D) = 7$, implying that x_4 has a neighbor in $X_{\leq 3}$, and each of x_5 and x_6 has a neighbor in $X_1 \cup X_2 \cup X_4$. Further, at least one of x_5 and x_6 has a neighbor in X_3 .

Since $x_5 \notin S_X$, the vertex x_5 is adjacent to some vertex x_j , where $j \in [4]$. If x_5 is adjacent to x_3 , then, noting that x_4 has a neighbor in $X_{\leq 3}$, the vertex x_5 and its two neighbors in $X_{\leq 4}$ belong to a common cycle of length at most 7, a contradiction. Hence, x_5 is not adjacent to x_3 and is therefore adjacent to x_1 , x_2 or x_4 . The girth condition implies now that x_5 has no neighbor in X_3 . Thus, x_6 has three neighbors in $X_{\leq 3}$, one in each of $X_1 \cup X_2 \cup X_4$, X_3 and X_5 . However, x_6 and its two neighbors in $X_{\leq 5} \setminus X_3$ belong to a common cycle of length at most 7, a contradiction. \square

By Claim 7.3.1, x_4 is adjacent to x_3 . By the girth condition, x_i has at most one neighbor in each of $X_1 \cup X_2$ and $X_3 \cup X_4$ for $i \in \{5, 6\}$, and so x_i has at most two neighbors in $X_{\leq 4}$. If x_5 has a neighbor in $X_1 \cup X_2$, then x_6 has at most one neighbor in $X_1 \cup X_2 \cup X_5$. If x_5 has a neighbor in $X_3 \cup X_4$, then x_6 has at most one neighbor in $X_3 \cup X_4 \cup X_5$. Hence, if x_5 has at least one neighbor in $X_{\leq 4}$, then x_6 has at most two neighbors in $X_{\leq 5}$. This implies that $m(D) \leq 7$. Analogously as in the proof of Claim 7.3.1, we deduce that $|S| = 6\delta - 11$, $|S_X^*| = 6\delta - 13$ and $S_X = \{x_1, x_3\}$. Further, $m(D) = 7$, implying that x_4 has a common neighbor with either x_1 or x_2 , and x_5 has two neighbors in $X_{\leq 4}$. Since $x_5 \notin S_X$, the vertex x_5 is adjacent to some vertex x_j , where $j \in [4]$. The above properties of the graph G imply that the vertex x_5 and its two neighbors in $X_{\leq 4}$ belong to a common cycle of length at most 7, a contradiction. This completes the proof of Claim 7.3. \square

By Claim 7.3, the vertex x_4 is not adjacent to x_1 , x_2 or x_3 . Hence by Claim 4(b), $x_4 \in S_X$, and so $|S_X| \geq 3$ and $\{x_1, x_3, x_4\} \subseteq S_X$.

Claim 7.4 *The vertex x_4 has no neighbor in $X_{\leq 3}$.*

Proof. Suppose, to the contrary, that x_4 has a neighbor in $X_{\leq 3}$. If x_4 has a neighbor in $X_1 \cup X_2$, then analogous arguments as in the proof of Claim 7.3.1 show that $m(D) \leq 7$. Hence, by Claim 2(c), $|S_X^*| \geq 6\delta - 13$. Thus, by Claim 2(a), $|S| \geq |S_X| + |S_X^*| \geq 6\delta - 10$, a contradiction. Hence, x_4 has no neighbor in $X_1 \cup X_2$, and therefore x_4 has a neighbor in X_3 . Thus, x_5 and x_6 have at most one neighbor in each of $X_1 \cup X_2$ and $X_3 \cup X_4$, implying that $m(D) \leq 7$ and therefore, as before, that $|S_X^*| \geq 6\delta - 13$ and $|S| \geq 6\delta - 10$, a contradiction. \square

We now return to the proof of Claim 7 one final time. By Claim 7.4, the vertex x_4 has no neighbor in $X_{\leq 3}$. Thus, by our earlier observation, the vertex x_4 is not adjacent to

x_1, x_2 or x_3 in the graph G_D . As observed earlier, x_1 and x_2 are adjacent in G_D , and x_3 is not adjacent to x_1 or x_2 in G_D . Since G_D is triangle-free, this implies that $m(D) \leq 7$. Hence, by Claim 2(c), $|S_X^*| \geq 6\delta - 13$. Thus, by Claim 2(a), $|S| \geq |S_X| + |S_X^*| \geq 6\delta - 10$, a contradiction. This completes the proof of Claim 7. (\square)

By Claim 7, the vertices x_1 and x_2 are not adjacent. Hence by Claim 4(b), $x_2 \in S_X$, implying that $\{x_1, x_2\} \subseteq S_X$ and $|S_X| \geq 2$.

Claim 8 *The vertex x_3 is not adjacent to x_1 or x_2 .*

Proof. Suppose, to the contrary, that x_3 is adjacent to x_1 or x_2 .

Claim 8.1 *The vertex x_3 is adjacent to exactly one of x_1 and x_2 .*

Proof. Suppose, to the contrary, that x_3 is adjacent to both x_1 and x_2 . In this case, x_2 has no neighbor in X_1 . Further, x_i has at most one neighbor in $X_{\leq 3}$ for $i \in \{4, \dots, g-2\}$. If $g = 7$, this implies that $m(D) \leq 5$, and so, by Claim 2(c), $|S_X^*| \geq 6\delta - 10$. Thus, by Claim 2(a), $|S| \geq |S_X| + |S_X^*| \geq 6\delta - 8$, a contradiction. Hence, $g = 8$, implying by the triangle-freeness of G_D and by our earlier observations that $m(D) \leq 7$. Thus, by Claim 2(c), $|S_X^*| \geq 6\delta - 13$. If x_4 is adjacent to some vertex x_j , where $j \in [3]$, then x_i has at most one neighbor in $X_{\leq 4}$ for $i \in \{5, 6\}$, implying that $m(D) \leq 6$, and so, by Claim 2(c), $|S_X^*| \geq 6\delta - 12$ and therefore, by Claim 2(a), $|S| \geq |S_X| + |S_X^*| \geq 6\delta - 10$, a contradiction. Hence, x_4 is not adjacent to some vertex x_j , where $j \in [3]$. Thus, by Claim 4(b), $x_4 \in S_X$, and so $|S_X| \geq 3$. Therefore, by Claim 2(a), $|S| \geq |S_X| + |S_X^*| \geq 3 + (6\delta - 13) = 6\delta - 10$, a contradiction. (\square)

By Claim 8.1, the vertex x_3 is adjacent to exactly one of x_1 and x_2 . Let $\{x_1, x_2\} = \{x_{i_1}, x_{i_2}\}$, where x_3 is adjacent to x_{i_1} . The girth requirement implies that each of x_{i_2} , x_4 , x_5 and, if $g = 8$, x_6 has at most one neighbor in $X_{i_1} \cup X_3$.

Claim 8.2 $g = 8$.

Proof. Suppose, to the contrary, that $g = 7$. As observed earlier, each of x_{i_2} , x_4 and x_5 has at most one neighbor in $X_{i_1} \cup X_3$. Therefore since G_D is triangle-free, $m(D) \leq 6$. Hence, by Claim 2(c), $|S_X^*| \geq 5\delta - 11$. Thus, by Claim 1 and Claim 2(a), $5\delta - 9 \geq |S| \geq |S_X| + |S_X^*| \geq 5\delta - 9$. Hence, we must have equality throughout this inequality chain, implying that $|S| = 5\delta - 9$, and so $|S_X| = 2$, $S_X = \{x_1, x_2\}$, and $|S_X^*| = 5\delta - 11$. Further, $m(D) = 6$, implying that each of x_{i_2} , x_4 and x_5 has a neighbor in $X_{i_1} \cup X_3$. Since $x_4 \notin S_X$, the vertex x_4 is adjacent to some vertex x_j , where $j \in [3]$. If x_4 is adjacent to x_{i_1} or x_3 , then x_4 has no neighbor in X_{i_2} , and x_5 has at most one neighbor in $X_{i_1} \cup X_3 \cup X_4$, implying that $m(D) \leq 5$, a contradiction. Hence, x_4

is adjacent to x_{i_2} . However, since both x_{i_2} and x_4 have a neighbor in $X_{i_1} \cup X_3$, we contradict the girth condition. \square

By Claim 8.2, $g = 8$.

Claim 8.3 *There is no edge in G_D joining x_{i_2} to x_{i_1} or x_3 .*

Proof. Suppose, to the contrary, that x_{i_2} is adjacent to x_{i_1} or x_3 in G_D . This implies that x_1 and x_2 have a common neighbor or x_3 has a common neighbor with x_{i_2} . Thus, x_i has at most one neighbor in $X_{\leq 3}$ for $i \in \{4, 5, 6\}$. Therefore since G_D is triangle-free, $m(D) \leq 7$. Hence, by Claim 2(c), $|S_X^*| \geq 6\delta - 13$. Thus, by Claim 1 and Claim 2(a), $6\delta - 11 \geq |S| \geq |S_X| + |S_X^*| \geq 6\delta - 11$. Hence, we must have equality throughout this inequality chain, implying that $|S| = 6\delta - 11$, and so $|S_X| = 2$, $S_X = \{x_1, x_2\}$, and $|S_X^*| = 5\delta - 13$. Further, $m(D) = 7$, implying that each of x_4 , x_5 and x_6 has at most one neighbor in $X_{\leq 3}$. Since $x_4 \notin S_X$, the vertex x_4 is adjacent to some vertex x_j , where $j \in [3]$. If x_4 is adjacent to x_{i_1} or x_3 , then x_4 has no neighbor in X_{i_2} , and both x_5 and x_6 have at most one neighbor in $X_{\leq 4}$, implying that $m(D) \leq 6$, a contradiction. Hence, x_4 is adjacent to x_{i_2} , and, by the girth condition, has no neighbor in $X_{i_1} \cup X_3$. Since $x_5 \notin S_X$, the vertex x_5 is adjacent to some vertex x_j , where $j \in [4]$, implying that x_5 has at exactly one neighbor in $X_{\leq 4}$ and x_6 has at most two neighbors in $X_{\leq 5}$. Once again, this implies that $m(D) \leq 6$, a contradiction. \square

By Claim 8.3, there is no edge in G_D joining x_{i_2} to x_{i_1} or x_3 . Thus, the vertices x_2 has no neighbor in X_1 , and the vertex x_3 has exactly one neighbor in $X_{\leq 2}$, namely the vertex x_{i_1} .

Claim 8.4 *The vertex x_4 is adjacent to no vertex x_j where $j \in [3]$.*

Proof. Suppose, to the contrary, that x_4 is adjacent to some vertex x_j , where $j \in [3]$. In this case, x_i has at most two neighbors in $X_{\leq 4}$ for $i \in \{5, 6\}$. Further, if x_5 has two neighbors in $X_{\leq 4}$, then x_6 has at most two neighbors in $X_{\leq 5}$, implying that $m(D) \leq 7$. If x_5 has at most one neighbor in $X_{\leq 4}$, then x_6 has at most three neighbors in $X_{\leq 5}$, implying once again $m(D) \leq 7$. Hence, $m(D) \leq 7$. Thus, by Claim 2(c), $|S_X^*| \geq 5\delta - 13$. Analogous arguments as before (see, for example, the proof of Claim 8.3) imply that $|S| = 6\delta - 11$, $S_X = \{x_1, x_2\}$, and $|S_X^*| = 5\delta - 13$. Further, $m(D) = 7$, implying in particular that x_4 has two neighbors in $X_{\leq 3}$. Since $x_5 \notin S_X$, the vertex x_5 is adjacent to some vertex x_j , where $j \in [4]$, implying that x_5 has at exactly one neighbor in $X_{\leq 4}$ and x_6 has at most two neighbors in $X_{\leq 5}$. This implies that $m(D) \leq 6$, a contradiction. \square

By Claim 8.4, the vertex x_4 is adjacent to no vertex x_j where $j \in [3]$. Hence by Claim 4(b), $x_4 \in S_X$, and so $\{x_1, x_2, x_4\} \subseteq S_X$. Thus, $|S_X| \geq 3$. Suppose that x_4 has at least one neighbor in $X_{i_1} \cup X_3$. The girth condition now implies that each of x_{i_2} , x_5 and x_6 has at most one neighbor in $X_{i_1} \cup X_3 \cup X_4$. Further, since G_D

is triangle-free, at most two of the edges $x_{i_2}x_5$, $x_{i_2}x_6$, and x_5x_6 are present in G_D . Thus, $m(D) \leq 7$. Hence, by Claim 2(c), $|S_X^*| \geq 5\delta - 13$. Thus, by Claim 2(a), $|S| \geq |S_X| + |S_X^*| \geq 3 + (6\delta - 13) \geq 6\delta - 10$, a contradiction. Therefore, x_4 has no neighbor in $X_{i_1} \cup X_3$. Hence, in G_D there is no edge joining a vertex in $\{x_{i_2}, x_4\}$ and a vertex in $\{x_{i_1}, x_3\}$. As observed earlier, each of x_5 and x_6 has at most one neighbor in $X_{i_1} \cup X_3$, and therefore there are at most two edges in G_D joining vertices in $\{x_5, x_6\}$ and vertices in $\{x_{i_1}, x_3\}$. Further since G_D is triangle-free, there are at most four edges in the subgraph of G_D induced by $\{x_{i_2}, x_4, x_5, x_6\}$. Therefore, $m(D) \leq 7$, once again producing a contradiction. This completes the proof of Claim 8. \square

By Claim 8, the vertex x_3 is not adjacent to x_1 or x_2 . By Claim 4(b), $x_3 \in S_X$, implying that $\{x_1, x_2, x_3\} \subseteq S_X$ and $|S_X| \geq 3$.

Claim 9 *The following holds.*

- (a) $g = 8$.
- (b) $S_X = \{x_1, x_2, x_3\}$.
- (c) $m(D) = 8$.

Proof. If $g = 7$, then by Claim 3, $|S_X^*| \geq 5\delta - 11$, and so by Claim 2(a), $|S| \geq |S_X^*| + |S_X| \geq 5\delta - 8$, a contradiction. Therefore, $g = 8$. This proves Part (a). We prove next that $|S_X^*| \geq 6\delta - 14$. Suppose, to the contrary, that $|S_X^*| < 6\delta - 14$. By Claim 3(b), this implies that $|S_X^*| = 6\delta - 15$. This in turn implies that the triangle-free graph G_D of order 6 has maximum possible size, namely 9, and is therefore the graph $K_{3,3}$. By Claim 1 and Claim 2(a), $6\delta - 11 \geq |S| \geq |S_X| + |S_X^*| = |S_X| + 6\delta - 15$, implying that $|S_X| \leq 4$. Since $\{x_1, x_2, x_3\} \subseteq S_X$, at least two vertices in $\{x_4, x_5, x_6\}$ do not belong to the set S_X . Let x_{i_0} be such a vertex in $\{x_4, x_5, x_6\}$. Thus, x_{i_0} is adjacent in G to some vertex x_{i_1} , where $i_1 < i_0$. Let $x_{i_0}x_{i_1}x_{i_2}x_{i_3}x_{i_0}$ be a 4-cycle in $G_D \cong K_{3,3}$ that contains the edge $x_{i_0}x_{i_1}$. We note that x_{i_j} and $x_{i_{j+1}}$ are either adjacent in G or have a common neighbor in G for $j \in \{1, 2, 3\}$, where addition is taken modulo 4. Thus, since $x_{i_0}x_{i_1}$ is an edge in G , this implies that x_{i_0} and x_{i_1} belong to a cycle of length at most 7 in G , a contradiction. Hence, $|S_X^*| \geq 6\delta - 14$. Thus, by Claim 1 and Claim 2(a), $6\delta - 11 \geq |S| \geq |S_X| + |S_X^*| \geq 3 + (6\delta - 14) = 6\delta - 11$. Hence, we must have equality throughout this inequality chain, implying that $|S| = 6\delta - 11$, and so $|S_X| = 3$, $S_X = \{x_1, x_2, x_3\}$. Further, $|S_X^*| = 5\delta - 14$, implying that $m(D) = 8$. This proves Parts (b) and (c). \square

Claim 10 *The vertex x_4 is adjacent to exactly one of x_1 , x_2 and x_3 .*

Proof. By Claim 9(b), the vertex $x_4 \notin S_X$, and so x_4 is adjacent to x_1 , x_2 or x_3 . Suppose, to the contrary, that x_4 is adjacent to at least two of x_1 , x_2 and x_3 . Let $\{x_1, x_2, x_3\} = \{x_{i_1}, x_{i_2}, x_{i_3}\}$, where x_4 is adjacent to x_{i_1} and x_{i_2} . The girth requirement implies that each of x_{i_3} , x_5 and x_6 has at most one neighbor in $X_{i_1} \cup X_{i_2} \cup X_4$. Further,

since G_D is triangle-free, at most two of the edges $x_{i_3}x_5$, $x_{i_3}x_6$, and x_5x_6 are present in G_D . Thus, $m(D) \leq 7$, contradicting Claim 9(c). \square

By Claim 10, the vertex x_4 is adjacent to exactly one of x_1 , x_2 and x_3 . Let $\{x_1, x_2, x_3\} = \{x_{i_1}, x_{i_2}, x_{i_3}\}$, where x_4 is adjacent to x_{i_1} . Suppose that x_{i_2} has a common neighbor with x_{i_1} or x_4 in G . The girth requirement implies that each of x_{i_3} , x_5 and x_6 has at most one neighbor in $X_{i_1} \cup X_{i_2} \cup X_4$. Further, since G_D is triangle-free, at most two of the edges $x_{i_3}x_5$, $x_{i_3}x_6$, and x_5x_6 are present in G_D . Thus, $m(D) \leq 7$, contradicting Claim 9(c). Hence, x_{i_2} has no common neighbor with x_{i_1} or x_4 in G . Analogously, x_{i_3} has no common neighbor with x_{i_1} or x_4 in G . Hence in G_D there is no edge joining a vertex in $\{x_{i_1}, x_4\}$ and a vertex in $\{x_{i_2}, x_3\}$. Each of x_5 and x_6 has at most one neighbor in $X_{i_1} \cup X_4$, and therefore there are at most two edges in G_D joining vertices in $\{x_5, x_6\}$ and vertices in $\{x_{i_1}, x_4\}$. Further since G_D is triangle-free, there are at most four edges in the subgraph of G_D induced by $\{x_{i_2}, x_{i_3}, x_5, x_6\}$. Therefore, $m(D) \leq 7$, contradicting Claim 9(c). This completes the proof of Theorem 2. \square

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